

(2nd revised version)

# Leafwise Symplectic Structures on Lawson's Foliation

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The aim of this paper is to show that Lawson's foliation on the 5-sphere admits a smooth leafwise symplectic structure. The main part of the construction is to show that the Fermat type cubic surface admits an end-periodic symplectic structure.

## 0 Introduction

In this article we show the following.

**Theorem A** (Theorem 3.1) Lawson's foliation on the 5-sphere  $S^5$  admits a smooth leafwise symplectic structure.

This work is totally motivated and inspired by the works ([MV1, MV2], [SV]) by Verjovsky and others in which they are discussing the existence of leafwise complex and symplectic structures on Lawson's foliations as well as on slightly modified ones. Especially, the author is extremely grateful to Alberto Verjovsky for drawing his attentions to such interesting problems.

H. B. Lawson, JR. constructed a smooth foliation of codimension one on  $S^5$  ([L]), which we now call *Lawson's foliation*. It was achieved by a beautiful combination of the complex and differential topologies and was a breakthrough in an early stage of the history of foliations. The foliation is composed of two components. One is a tubular neighbourhood of a 3-dimensional nil-manifold and the other one is, away from the boundary, foliated by Fermat-type cubic complex surfaces. As the common boundary leaf, here appears one of Kodaira-Thurston's 4-dimensional nil-manifolds. As each Fermat cubic leaf is spiraling to this boundary leaf, its end is diffeomorphic to a cyclic covering of Kodaira-Thurston's nil-manifold. (See Section 1 for the detail.)

In order to introduce a leafwise symplectic structure (for a precise definition, see Section 2), at least, we have to find a symplectic structure on the Fermat cubic surface which (asymptotically) coincides on the end with that of the cyclic covering of the Kodaira-Thurston nil-manifold. However,

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the natural symplectic structure on the Fermat cubic surface as a Stein surface is quite different from the periodic ones on its end, because it is ‘conic’ and expanding and therefore not periodic. This is the crucial point in our problem. Once we find an end-periodic symplectic structure on the Fermat cubic surface (Section 5), in fact that is almost enough to construct a smooth leafwise symplectic structure (Section 3), because it is easy to see that a natural foliation on the tube component admits a leafwise symplectic structure (Section 2).

We can paraphrase our main result into the following.

**Corollary B** The 5-sphere  $S^5$  admits a regular Poisson structure of symplectic dimension 4 whose symplectic foliation exactly coincides with Lawson’s one.

This paper is organized as follows. Lawson’s foliation is reviewed in Section 1. In Section 2, a leafwise symplectic structure on the tube component is given. Also symplectic structures on Kodaira-Thurston’s nil-manifold and its covering are presented. This enables us to state our method precisely in Section 3. Assuming the results on symplectic structures on the Fermat cubic surface in the later sections, a construction of a leafwise symplectic structure on Lawson’s foliation is given in this section. Then, the natural symplectic structure on the Fermat cubic surface is analyzed in Section 4. Based on this analysis, in Section 5, which is the essential part of the present article, the existence of an end-periodic symplectic structure on the Fermat cubic surface is shown. In Section 6 we remark that our construction holds almost verbatim in two cases where we replace the Fermat cubic polynomial with the polynomials related to the simple elliptic hypersurface singularities. In the final section, some related topics concerning the method in this paper are discussed.

Most pages are devoted to describe Lawson’s foliation and the natural symplectic structure on the Fermat cubic surfaces. Really new items appear only in Section 5 where we see one of the fantastic and mysterious natures that the 4-dimensional spaces have against symplectic structures.

## 1 Review of Lawson’s Foliation

First we review the structure of Lawson’s foliation  $\mathcal{L}$ . For those who are familiar with the materials it is enough to check our notations, which are in fact quite different from [MV1, MV2] and even from [L].

Let us take a Fermat type homogeneous cubic polynomial  $f(Z_0, Z_1, Z_2) = Z_0^3 + Z_1^3 + Z_2^3$  in three variables  $Z_0, Z_1$ , and  $Z_2$ . The complex surface  $F_w = \{(Z_0, Z_1, Z_2) \in \mathbb{C}^3; f(Z_0, Z_1, Z_2) = w\}$  for a complex value  $w$  is non-singular if  $w \neq 0$  and  $F_0$  has the unique singularity at the origin. The scalar multiplication  $c \cdot (Z_0, Z_1, Z_2) = (cZ_0, cZ_1, cZ_2)$  by  $c \in \mathbb{C}$  maps  $F_w$  to  $F_{c^3w}$ . Hence  $F_0$  is preserved by such homotheties and  $F_w$  for  $w \neq 0$  is

preserved iff  $c^3 = 1$ .

Now we put  $\tilde{\Lambda}_\theta = \bigcup_{\arg w = \theta} F_w$  and  $\Lambda_\theta = \tilde{\Lambda}_\theta \cap S^5$  where  $S^5$  denotes the unit sphere in  $\mathbb{C}^3$ . Also, we put  $N = \text{Nil}^3(-3) = F_0 \cap S^5$ . Let  $p$  and  $h$  denote the projection  $p : \check{\mathbb{C}}^3 \rightarrow S^5$  and the Hopf fibration  $h : S^5 \rightarrow \mathbb{CP}^2$ . Here  $\check{\mathbb{C}}^3$  denotes  $\mathbb{C}^3 \setminus \{O\}$ . Sometimes  $h$  also denotes the composition  $h \circ p : \check{\mathbb{C}}^3 \rightarrow \mathbb{CP}^2$ . Let also  $H(t)$  ( $t \in \mathbb{R}$ ) denote the *Hopf flow* obtained by scalar multiplication by  $e^{it}$ , whose orbits are the Hopf fibres.  $E_\omega = \{[z_0 : z_1 : z_2]; z_0 + z_1 + z_2 = 0\} = h(F_0)$  is an elliptic curve in  $\mathbb{CP}^2$  with modulus  $\omega = \frac{-1+\sqrt{-3}}{2}$ . The Hopf fibration restricts to  $N \rightarrow E_\omega$ , which is an  $S^1$ -bundle with  $c_1 = -3$ . The following facts are also easy to see, while they are listed as Proposition for the sake of later use.

- Proposition 1.1** 1)  $f|_{S^5}$  has no critical points around  $N$ .  
2)  $\arg \circ f|_{S^5 \setminus N} : S^5 \setminus N \rightarrow S^1$  has no critical points away from  $N$  and is called the *Milnor fibration*. Each fibre is by  $\Lambda_\theta$  ( $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ ).  
3)  $p|_{F_w} : F_w \rightarrow \Lambda_\theta$  ( $\theta = \arg w$ ) is a diffeomorphism for  $w \neq 0$ .  
4) The Hopf fibration  $h|_{\Lambda_\theta} : \Lambda_\theta \rightarrow \mathbb{CP}^2 \setminus E_\omega$  restricted to  $\Lambda_\theta$  is a three fold regular covering, and so is  $F_w \rightarrow \mathbb{CP}^2 \setminus E_\omega$  for  $w \neq 0$ .  
5) The normal bundle to  $N \hookrightarrow S^5$  is trivialized by the value of  $f$ .  
6)  $H(2\pi/3)$  gives the natural monodromy of the Milnor fibration.

Take a tubular neighbourhoods  $W_\varepsilon$  of  $N \subset S^5$  and  $U_\varepsilon$  of  $E_\omega \subset \mathbb{CP}^2$  as  $W_\varepsilon = \{Z = (Z_0, Z_1, Z_2) \in S^5; |f(Z)| \leq \varepsilon\}$  and  $U_\varepsilon = h(W_\varepsilon) \subset \mathbb{CP}^2$  for a small  $\varepsilon$  ( $0 < \varepsilon \ll 1$ ) such that  $f|_{W_\varepsilon}$  has no critical points.  $W_\varepsilon$  is invariant under the Hopf flow. We choose further smaller constants  $r_0$  and  $r_*$  satisfying  $0 < r_0 < r_* < \varepsilon$  and take  $W_* = W_{r_*}$ ,  $W = W_{r_0}$  and  $U_* = U_{r_*}$ ,  $U = U_{r_0}$ . We decompose  $S^5$  into  $W$  and  $C = S^5 \setminus \text{Int}W$ , each of which are called the *tube* component and the *Fermat cubic* component. The statement 5) in the above proposition tells that  $W_r$  is diffeomorphic to the product  $N \times D_r^2$ , while  $E_\omega \hookrightarrow \mathbb{CP}^2$  is twisted because  $[E_\omega]^2 = 9$ . Here  $D_r^2$  denotes the disk of radius  $r$  in  $\mathbb{C}$ .

The common boundary  $\partial W = \partial C$  is diffeomorphic to  $N \times S^1$ , which is one of Kodaira-Thurston's 4-dimensional nil-manifolds and is well-known to be non-Kähler because  $b_1 = 3$ . It admits a symplectic structure and a complex structure independently but both are never compatible.

As the two components are fibering over the circle, the following lemma (a standard process of *turbulization*) is enough only to obtain a smooth foliation. However, to put leafwise symplectic structures, it is helpful to describe the foliation and the turbulization in more detail.

**Lemma 1.2** ([L], Lemma 1) Let  $M$  be a compact smooth manifold with boundary  $\partial M$  and  $\varphi : M \rightarrow S^1$  be a smooth submersion to the circle. Accordingly so is  $\varphi|_{\partial M} : \partial M \rightarrow S^1$ . Then, there exists a smooth foliation of codimension one for which the boundary  $\partial M$  is the unique compact leaf,

other leaves are diffeomorphic to the interior of the fibres, and the holonomy of the compact leaf is trivial as a  $C^\infty$ -jet. If we have two such submersions  $\varphi_i : M_i \rightarrow S^1$  ( $i = 1, 2$ ) with diffeomorphic boundaries  $\partial M_1 \cong \partial M_2$ , then on the closed manifold  $M_1 \cup_{\partial M_1 = \partial M_2} M_2$ , by gluing them we obtain a smooth foliation of codimension one.

Let us formulate the turbulization process more explicitly. On  $\mathbb{R}^2 \setminus \{O\} = \mathbb{R}_+ \times \mathbb{R} \ni (r, \theta)$ , take small constants  $0 < r_0 < r_1 < r_2 < r_*$  and smooth functions  $g(r)$  and  $h(r)$  satisfying the following conditions with  $c = \frac{\log \pi}{3}$  and define a smooth non-singular vector field  $X = g \frac{\partial}{\partial r} + h \frac{\partial}{\partial \theta}$ .

$$\begin{aligned} g &\equiv 0 & (r \leq r_0), & & h &\equiv 1 & (r \leq r_1), \\ g &= -cr & (r_1 \leq r < r_*), & & h &\equiv 0 & (r_2 \leq r < r_*), \\ g' &< 0 & (r_0 < r < r_*), & & h' &< 0 & (r_1 < r < r_2). \end{aligned}$$

The integral curves of  $X$  define a smooth foliation  $\mathcal{F}_T$  on the right half plane  $\mathbb{R}_+^2$  as well as on the punctured plane  $\mathbb{C} \setminus \{O\} \cong \mathbb{R}_+ \times S^1$  where the second factor  $S^1$  is considered to be  $\mathbb{R}/2\pi\mathbb{Z}$  and  $(r, \theta)$  denotes the polar coordinates. The constant  $\frac{\log \pi}{3}$  has no significance at this stage.

Now the turbulization on the side of the Fermat component is described as follows. The foliation  $\tilde{\mathcal{L}} = \{\Lambda_\theta\}$  on  $S^5 \setminus N$  by the Milnor fibres and the pull-back foliation  $f^{-1}\mathcal{F}_T$  coincide with each other on  $W_* \setminus W_{r_2}$ . On the Fermat component  $C = S^5 \setminus \text{Int } W$ , Lawson's foliation  $\mathcal{L}|_C$  is obtained as  $\mathcal{L}|_{S^5 \setminus W_{r_2}} = \tilde{\mathcal{L}}|_{S^5 \setminus W_{r_2}}$  and  $\mathcal{L}|_{W_* \setminus W} = f^{-1}\mathcal{F}_T|_{W_* \setminus W}$ . Let  $L_\theta$  denote one of the resulting leaves which contains  $\Lambda_\theta \setminus W_{r_2}$ .  $L_\theta$  is diffeomorphic to  $\Lambda_\theta$  and only the embedding of the product end  $N \times \{r \cdot e^{i\theta}\}$  is modified by the turbulization procedure. We will fix an identification of  $L_\theta$  with  $F_{e^{i\theta}}$  in Section 3.

On the tube component, we can also formulate the turbulization using  $\mathcal{F}_T$  in a similar way to the above, however, since  $\mathcal{L}|_W$  can also be described using a well-known foliation, the "Reeb component", on  $S^1 \times D^2$ , here we adopt such a description. The tube component  $W$  is diffeomorphic to  $N \times D^2$  and  $N = \text{Nil}^3(-3)$  is an  $S^1$ -bundle over the elliptic curve  $E_\omega$ . We take a smooth coordinate  $(x, y)$  for  $E_\omega$  where  $x, y \in S^1 = \mathbb{R}/2\pi\mathbb{Z}$ . Then the projection from  $E_\omega$  to  $S^1 \ni x$  gives rise to a fibration of  $N$  over  $S^1$  with fibre  $T^2$  and the monodromy  $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ . Therefore the tube component  $W$  fibers over the solid torus  $S^1 \times D^2$  with the fibre  $T^2$  and the monodromy  $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ , namely,  $W = \mathbb{R} \times D^2 \times T^2 / \sim$  where  $(x + 2\pi, P, \begin{pmatrix} y \\ z \end{pmatrix}) \sim (x, P, \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y \\ z \end{pmatrix})$ .

As the tube component part of Lawson's foliation  $\mathcal{L}$ , we can take the pull-back of the standard Reeb component  $\mathcal{F}_R$  on  $S^1 \times D^2$  to  $W$ . Thus we obtain Lawson's foliation  $\mathcal{L}$  on  $S^5$ , whose boundary is a unique compact leaf and is diffeomorphic to Kodaira-Thurston's nil-manifold.

**Remark 1.3** The 3-dimensional nil-manifold  $\text{Nil}^3(c_1)$  is often presented as the quotient  $\text{Nil}^3(c_1) = \Gamma(c_1) \setminus H$  of the 3-dimensional Heisenberg group  $H$  by its lattice  $\Gamma(c_1)$ , which are defined as

$$H = \left\{ \begin{pmatrix} 1 & \bar{x} & \bar{z} \\ 0 & 1 & \bar{y} \\ 0 & 0 & 1 \end{pmatrix} ; \bar{x}, \bar{y}, \bar{z} \in \mathbb{R} \right\} \supset \Gamma(c_1) = \left\{ \begin{pmatrix} 1 & \bar{x} & \bar{z} \\ 0 & 1 & \bar{y} \\ 0 & 0 & 1 \end{pmatrix} ; \bar{x}, \bar{y}, c_1 \bar{z} \in \mathbb{Z} \right\}.$$

In the case  $c_1 < 0$ ,  $\bar{z}$  must be understood to have opposite sign. In this coordinate on  $H$  take  $\frac{\partial}{\partial \bar{x}}$ ,  $\frac{\partial}{\partial \bar{y}}$ , and  $\frac{\partial}{\partial \bar{z}}$  at the unit element, and then extend them to be  $X$ ,  $Y$ , and  $Z$  as left invariant vector fields. Let  $d\bar{x}$ ,  $d\bar{y}$ , and  $\bar{\zeta}$  be the dual basis for the invariant 1-forms, which satisfies  $d\bar{\zeta} = d\bar{x} \wedge d\bar{y}$ . On our  $N = Nil^3(-3)$  we have  $x = 2\pi\bar{x}$ ,  $y = 2\pi\bar{y}$ ,  $z = 2\pi c_1 \bar{z}$ , and  $\zeta = 2\pi c_1 \bar{\zeta}$ .

**Remark 1.4** As the normal bundle to  $E_\omega \hookrightarrow \mathbb{CP}^2$  has  $c_1 = 9$ , the boundary  $\partial U$  is isomorphic to  $Nil^3(9)$ . For this we look at  $\partial U$  from the interior of  $U$ . However, for the later purpose, it is more convenient to give the opposite orientation, because to  $N$  we gave the orientation as the boundary of  $F_0 \cap \{\rho \leq R\}$  and this implies also as the end of  $F_1$ . Therefore let  $N'$  denote  $\partial(\mathbb{CP}^2 \setminus U)$ , which is isomorphic to  $Nil^3(-9)$ .

## 2 Symplectic Forms on the Kodaira-Thurston Nil-Manifold and $F_0$

In this section, we describe natural symplectic forms on Kodaira-Thurston's 4-dimensional nil-manifold and show that the tube component admits a smooth leafwise symplectic structure which is *tame* around the boundary.

**Definition 2.1** A smooth *leafwise symplectic structure* (or *form*) on a smooth foliated manifold  $(M, \mathcal{F})$  is a smooth leafwise closed 2-form  $\beta$  which is non-degenerate on each leaves.

More precisely, first,  $\beta$  is a smooth section to the smooth vector bundle  $\wedge^2 T^* \mathcal{F}$ . For smooth sections to  $\wedge^* T^* \mathcal{F}$  naturally the exterior differential in each leaves is defined. This exterior differential is often denoted by  $d_{\mathcal{F}}$ .  $\beta$  is closed in this sense and is non-degenerate in each leaves, namely,  $d_{\mathcal{F}} \beta = 0$  holds and  $\beta^{\dim \mathcal{F}/2}$  defines a volume form on each leaves.

The existence of such  $\beta$  is equivalent to that of a smooth 2-form  $\tilde{\beta}$  on  $M$  whose restriction to each leaf is a symplectic form of the leaf. It should be remarked that  $\tilde{\beta}$  may not be closed as a 2-form on  $M$ . Furthermore, quite often we mix the two formulations and do not make a clear distinction.

**Definition 2.2** Let  $(M, \mathcal{F})$  be a smooth foliated manifold with a boundary compact leaf  $\partial M$  and a leafwise symplectic form  $\beta$ .  $(M, \mathcal{F}, \beta)$  is *tame* around the boundary if the triple satisfies the following condition. We also simply say that  $\beta$  is tame.

- (1) The (one-sided) holonomy of the boundary leaf is trivial as  $C^\infty$ -jet.
- (2) There exists a collar neighbourhood  $V \cong [0, \epsilon) \times \partial M$  of the boundary  $\partial M$  with the projection  $Pr : [0, \epsilon) \times \partial M \rightarrow \partial M$  for which  $\beta|_V$  coincides with the restriction to the leaves of the pull-back  $Pr^*(\beta|_{\partial M})$ .

**Corollary 2.3** Let  $(M_i, \mathcal{F}_i, \beta_i)$  ( $i = 0, 1$ ) be two foliated manifolds with leafwise symplectic structures. Assume that both are tame around their boundaries and there exists a symplectomorphism  $\varphi : (\partial M_1, \beta_1|_{\partial M_1}) \rightarrow (\partial M_2, \beta_2|_{\partial M_2})$  between their boundaries. Then gluing by  $\varphi$  yields a smooth foliated manifold  $(M = M_1 \cup_\varphi M_2, \mathcal{F}, \beta)$  with a smooth leafwise symplectic structure.

For the Reeb component  $(S^1 \times D^2, \mathcal{F}_R)$  it is easy to show that there exists a tame leafwise symplectic structure. First fix an area form of the boundary, then extend it to a collar neighbourhood by the tameness condition, and finally further extend it to a leafwise 2-form such that on each leaves it gives area forms.

Combined with the description of the tube component in the end of the previous section, this observation enables us easily to show that the tube component admits a tame leafwise symplectic structure as follows. Take a coordinate  $(x, r, \theta)$  for the solid torus  $S^1 \times D^2 = \{(x, r, \theta) ; x \in S^1 = \mathbb{R}/2\pi\mathbb{Z}, 0 \leq r \leq r_0, \theta \in S^1 = \mathbb{R}/2\pi\mathbb{Z}\}$ . Let  $\zeta$  denote the standard connection 1-form for the Hopf fibration  $h : S^5 \rightarrow \mathbb{C}P^2$ .  $\zeta$  coincides with the standard contact form  $\zeta = \sum_{j=1}^3 (x_j dy_j - y_j dx_j)$  on  $S^5$ . On each fibre (with an arbitrary reference point)  $\zeta$  defines an identification with  $S^1 = \mathbb{R}/2\pi\mathbb{Z}$  and the resulting coordinate is denoted by  $z$  in the previous section. Once  $\zeta$  is restricted to  $N = F \cap S^5$  it is denoted by  $\zeta_N$ .

The tube component  $W$  admits a flat bundle structure  $T^2 \hookrightarrow W = N \times D^2 \xrightarrow{\pi_R} S^1 \times D^2$  with the monodromy  $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$ . On  $N$  we have  $d\zeta_N = -(-3)2\pi \frac{dx}{2\pi} \wedge \frac{dy}{2\pi} = \frac{3}{2\pi} dx \wedge dy$  and hence  $dy \wedge \zeta_N$  is a closed 2-form which restricts to each fibre  $\cong T^2$  to be a holonomy invariant area form.

On the other hand, starting with the standard area form  $d\theta \wedge dx$  on the boundary of the Reeb component, we can equip the Reeb component with a tame leafwise symplectic form  $\beta_R$ . As Lawson's foliation on the tube component is given as  $\pi_R^{-1}\mathcal{F}_R$ , using the pull-back  $\pi_R^*\beta_R$  we obtain a tame leafwise symplectic form  $\beta_{W,\lambda,\mu} = \lambda \pi_R^*\beta_R + \mu dy \wedge \zeta_N$  for non-zero constants  $\lambda$  and  $\mu$ . The restriction of  $\beta_{W,\lambda,\mu}$  to the boundary  $\partial W$  is presented as  $\lambda d\theta \wedge dx + \mu dy \wedge \zeta_N$ . Also it is easy to see that the foliation  $\mathcal{L}|_W$  (in fact  $\mathcal{L}$  itself) and the leafwise symplectic form  $\beta_{W,\lambda,\mu}$  is invariant under the Hopf flow  $H(t)$ . We have established the following.

**Proposition 2.4** On the tube component Lawson's foliation  $\mathcal{L}|_W$  admits a tame leafwise symplectic form  $\beta_{W,\lambda,\mu} = \lambda \pi_R^*\beta_R + \mu dy \wedge \zeta_N$  for constants  $\lambda \neq 0$  and  $\mu \neq 0$ , which is invariant under the Hopf flow. It restricts to  $\beta_{K,\lambda,\mu} = \lambda d\theta \wedge dx + \mu dy \wedge \zeta_N$  on the boundary leaf.

In order to make a better correspondence with the Fermat component, we introduce a new coordinate variable  $\tau$  which replaces  $\theta$  on Kodaira-Thurston's nil-manifold in these symplectic forms.  $\theta$  is reserved for the argument of the value of  $f$ . Then consider a new relation  $\tau = 2 \log \rho$

between the radius  $\rho = \sqrt{|Z_0|^2 + |Z_1|^2 + |Z_2|^2}$  on  $\mathbb{C}^3$ , which fits into the following picture. Since our Kodaira-Thurston nil-manifold  $K$  is presented as  $K = \partial W = N \times S^1$  and also the polar coordinate  $\mathbb{R}_+ \times S^5$  for  $\check{\mathbb{C}}^3$  restricts to  $\check{F}_0 = F_0 \setminus \{O\} = \mathbb{R}_+ \times N$ , we identify  $K$  with  $\check{F}_0/\sim$  where  $P \sim Q$  for  $P$  and  $Q \in \check{F}_0$  iff  $Q = e^{n\pi}P$  for some  $n \in \mathbb{Z}$ .

**Remark 2.5** Instead of  $e^\pi$ , we can take any complex number  $c \in \mathbb{C}$  with  $|c| > 1$ . Then according to  $c$  the complex structure induced on  $K$  from  $F_0$  varies. However, they are all diffeomorphic to each other and in this article we are rather interested in the symplectic structure of  $K$ .

On the Fermat component  $C$ , any of the interior leaf is diffeomorphic to the Fermat cubic surface  $F_1$ , and is spiraling to the boundary leaf  $K$ . The way in which an interior leaf approaches to  $K$  is topologically the same as  $F_1$  approaches to  $F_0$  after divided by the scalar multiplication by  $e^{n\pi}$  for  $\forall n \in \mathbb{Z}$ . Looking at  $\check{F}_0$  and  $F_{e^{i\theta}}$ 's in this way is used in later sections. It is also helpful in introducing leafwise complex structures on the Fermat component. In the rest of this section, we take a closer look at  $K = \check{F}_0/\sim$  and  $\check{F}_0 = \mathbb{R}_+ \times N$ .

From  $K$  the infinite cyclic covering pulls  $\beta_{K,\lambda,\mu}$  back to a periodic symplectic form  $\beta_{0,\lambda,\mu} = \lambda d\tau \wedge dx + \mu dy \wedge \zeta_N = \lambda \frac{2}{\rho} d\rho \wedge dx + \mu dy \wedge \zeta_N$  on  $\check{F}_0$ . On the other hand,  $\check{F}_0$  inherits a natural symplectic structure from  $(\mathbb{C}^3, \beta^* = 2 \sum_{j=0}^2 dx_j \wedge dy_j)$ . One of the standard Liouville forms (the primitive of symplectic form)  $\lambda^* = \sum_{j=0}^2 (x_j dy_j - y_j dx_j)$  is presented as  $\rho^2 \zeta_N$  in the polar coordinate. Replacing  $\rho^2$  with  $\bar{\rho}$ , we see that  $(\check{F}_0, \beta_0 = d(\bar{\rho} \zeta_N))$  is the symplectization of the contact manifold  $(N, \zeta_N)$ . Replacing  $\rho$  with  $\tau = \log \rho = 2 \log \rho \in \mathbb{R}$ , we have an identification of  $\check{F}_0$  with  $N \times \mathbb{R} \ni (P, \tau)$ . We call this the *product* coordinate.

In the next section, the product coordinate is also defined on the end of  $F_1$ .

### 3 Smooth Leafwise Symplectic Structure on Lawson's Foliation

#### 3.1 Main theorems and coordinates

**Theorem 3.1** Lawson's foliation  $\mathcal{L}$  on  $S^5$  admits a smooth leafwise symplectic form.

This is the main theorem of the present article. Proposition 2.4 and Corollary 2.3 imply that this is a direct consequence of the following proposition, which we prove in this section assuming Corollary 5.2.

**Proposition 3.2** For a sufficiently small constant  $0 < \mu \ll 1$  and a sufficiently large constant  $\lambda \gg 1$  Lawson's foliation restricted to the Fermat

component admits a tame symplectic form which restricts to  $\beta_{K,\lambda,\mu} = \lambda d\tau \wedge dx + \mu dy \wedge \zeta_N$  on the boundary leaf  $K$ .

As a preparation, we start with a more precise description of tubular neighbourhoods of  $N$  in  $S^5$  and those of  $\check{F}_0$  in  $\check{C}^3$ . For the tubular neighbourhood  $W_\varepsilon$  of  $N$  in  $S^5$ , we describe an identification of  $W_\varepsilon$  with  $N \times D_\varepsilon^2$ . As  $f : W_\varepsilon \rightarrow D_\varepsilon^2 \subset \mathbb{C}$  defines the second projection, it is enough to define the projection to  $N$ .

First decompose  $T_P S^5 = T_P N \oplus (T_P N)^\perp$  at each point  $P \in N$  with respect to the standard metric. Then using the exponential maps, we can define in a canonical way a fibration of a small tubular neighbourhood of  $N$  to  $N$ .

The Hopf action  $H(t)$  and the complex conjugation  $\mathcal{C} : \mathbb{C}^3 \rightarrow \mathbb{C}^3$  ( $\mathcal{C}(Z_0, Z_1, Z_2) = (\bar{Z}_0, \bar{Z}_1, \bar{Z}_2)$ ) act as isometries on  $\mathbb{C}^3$  and  $\mathcal{C}$  sends the Milnor fibre  $\Lambda_\theta$  to  $\Lambda_{-\theta}$ . Therefore the smooth submanifolds  $\tilde{\Lambda}_\theta = \Lambda_\theta \cup N \cup \Lambda_{-\theta}$  ( $\theta \in S^1$ ) in  $S^5$  are all totally geodesic. The geodesics which define the above projection  $W_\varepsilon \rightarrow N$  go inside each  $\tilde{\Lambda}_\theta$ . This means that through the identification,  $N \times (0, \varepsilon) \times \{\theta\}$  coincides exactly with  $\Lambda_\theta \cap W_\varepsilon$ .

We can do the same on  $\rho \cdot N (= \{\rho\} \times N$  in the polar coordinate) in the sphere  $S^5(\rho)$  of the radius  $\rho$ . The tubular neighbourhood  $\rho \cdot W_\varepsilon$  of  $\rho \cdot N$  has the similar identification with  $N \times D_\varepsilon^2$ . The projection to  $N$  is defined in the same way. We define the second projection to  $D_\varepsilon^2$  not by  $f$  but by  $\rho^{-3}f$  so as to make it invariant under the homotheties by multiplying positive real numbers. Collecting these identifications with respect to each  $\rho > 0$ , it also defines an identification of the tubular neighbourhood  $\check{C}^3 \cap \{|\rho^{-3}f| < \varepsilon\}$  of  $\check{F}_0$  ( $0 < \varepsilon \ll 1$ ) with  $\check{F}_0 \times D_\varepsilon^2$  or with  $\mathbb{R}_+ \times N \times D_\varepsilon^2$ . The end  $F_1 \cap \{\rho > \varepsilon^{-\frac{1}{3}}\}$  [*resp.*  $F_{e^{i\theta}} \cap \{\rho > \varepsilon^{-\frac{1}{3}}\}$ ] of the Fermat cubic surface  $F_1$  [*resp.*  $F_{e^{i\theta}}$ ] is exactly the graph of the function  $\rho^{-3}$  [*resp.*  $\rho^{-3}e^{i\theta}$ ]. From this we see that these ends are diffeomorphic to  $(\varepsilon^{-\frac{1}{3}}, \infty) \times N$ , which is called the *product coordinate* of the end. Later,  $\tau = 2 \log \rho$  replaces  $\rho$ , where the identification is denoted by  $N \times (-\frac{2}{3} \log \varepsilon, \infty) \ni (P, \tau)$  and is also called the product coordinate.

**Remark 3.3** This identification is equivariant under the Hopf flow  $H(t)$  if we interpret the action of  $H(t)$  on  $\mathbb{R}_+ \times N \times D_\varepsilon^2$  is  $(\rho, P, z) \mapsto (\rho, H(t) \cdot P, e^{i3t}z)$  where  $H(t) \cdot P$  is the restriction of the Hopf action on  $N \subset S^5$  and is nothing but the fibrewise multiplication by  $e^{it}$  on each fibre of  $S^1 \hookrightarrow N \rightarrow E_\omega$ . Similarly, on  $W_\varepsilon \setminus N (\cong N \times D_\varepsilon^2) \cong N \times (0, \varepsilon) \times S^1$ , the Hopf action is indicated as  $H(t)(P, r, \theta) = (H(t)(P), r, \theta + 3t)$ .

### 3.2 Proof of main proposition

First let us introduce a leafwise symplectic structure on the Milnor fibration  $(S^5 \setminus N, \{\Lambda_\theta\})$ . Following Corollary 5.2, take and fix a symplectic form  $\beta_{\lambda,\mu}$  on  $F_1$ . Each leaf  $\Lambda_\theta$  is identified with  $F_1$  through the projec-



tion and the Hopf actions;  $F_1 \xrightarrow{p} \Lambda_0 \xrightarrow{H(t)} \Lambda_\theta$  for  $t = \frac{\theta+2k\pi}{3} \pmod{2\pi\mathbb{Z}}$  ( $k = 0, 1, 2$ ). Let  $p_{\theta,k} : F_1 \rightarrow \Lambda_\theta$  denote this identification. For the sake of continuity we can not decide which one to choose among three values of  $t$ . However as  $\beta_{\lambda,\mu}$  is invariant under the action of  $H(\frac{2\pi}{3}\mathbb{Z})$  on  $F_1$ , these indentifications induce a well-defined symplectic form on each  $\Lambda_\theta$  from  $(F_1, \beta_{\lambda,\mu})$ , which gives rise to a smooth leafwise symplectic structure  $\beta_{\tilde{\mathcal{L}}}$  on the Milnor fibration.

Next we go back to the turbulizing process of obtaining the Fermat component. Here we need a pointwise identification of each interior leaf  $L_\theta$  of the Fermat component with  $\Lambda_\theta$  and with  $\Lambda_0$ . On  $W_* \setminus N \cong N \times (D_{r_*}^2 \setminus \{O\})$ , take a vector field  $\tilde{X}$  which is defined as  $(0, X)$ , where  $X$  is the vector field defined on  $D_{r_*}^2 \setminus \{O\}$  in Section 1 for the turbulization. Also take a vector field  $\tilde{R} = (0, -\frac{\log \pi}{3} r \frac{\partial}{\partial r})$ .

We identify  $L_0$  with  $\Lambda_0$  and thus with  $F_1$  as follows. The core part  $L_0 \setminus W_{r_2}$  is exactly identical with  $\Lambda_0 \setminus W_{r_2}$ . For  $t > -\frac{3}{\log \pi}(\log r_* - \log r_2)$ , the point  $\exp(t\tilde{X})(P, r_2, 0)$  of  $L_0$  is identified with the point  $\exp(t\tilde{R})(P, r_2, 0)$  of  $\Lambda_0$ . Accordingly, these points are identified with the point  $(P, \tau_2 + t)$  of  $F_1$  in the product coordinate, where  $\tau_2 = -\frac{2}{3} \log r_2$ . Similarly  $\exp(t\tilde{X})(P, r_2, \theta) \in L_\theta$  and  $\exp(t\tilde{R})(P, r_2, \theta) \in \Lambda_\theta$  are identified. Let  $lm_\theta : \Lambda_\theta \rightarrow L_\theta$  denote this identification. Remark that through these identifications a point  $(P, \tau)$  of the end of  $F_1$  for large enough  $\tau \gg 0$  is sent to a point  $lm_0 \circ p(P, \tau) = (P, r(\tau), \tau + c_0)$  in  $L_0 \cap W_{r_2}$  for some function  $r(\tau)$  and some constant  $c_0$ . Also we have  $lm_\theta \circ p_{\theta,k}(P, \tau) = (H(\frac{\theta+2k\pi}{3})(P), r(\tau), \tau + c_0 + \theta)$ .

The leafwise symplectic form  $\beta_{\tilde{\mathcal{L}}}$  on the Milnor fibration is thus transplanted on the interior of the Fermat component  $C$  of Lawson's foliation  $\mathcal{L}$  to be a leafwise symplectic form  $\beta_C$ . What remains to prove is that  $\beta_C|_{W_{r_1} \setminus \overline{W}}$  coincides with  $Pr^* \beta_{K,\lambda,\mu}$  where  $Pr$  denotes the projection of the end  $W_{r_1} \setminus \overline{W} \cong N \times (r_0, r_1) \times S^1$  of  $C$  to the boundary  $\partial C \cong N \times \{r_0\} \times S^1 \cong N \times S^1 (= K)$ . Then we obtain a tame symplectic form on  $C$  with the restriction  $\beta_{K,\lambda,\mu}$  to the boundary, so that Corollary 2.3 proves our main proposition.

From the above preparations, we see that the composition of the maps  $Pr \circ lm_\theta \circ p_{\theta,k} : [\text{the end of } F_1] = N \times (T, \infty) \rightarrow \partial C = N \times S^1$  sends the points as  $(P, \tau) \mapsto (H(\frac{\theta+2k\pi}{3})(P), \tau + c_0 + \theta)$  for some  $T \gg 0$ . As is mentioned in Corollary 5.2,  $\beta_{\lambda,\mu}|_{N \times (T, \infty)}$  is invariant under the Hopf action and the  $\tau$ -translation for any  $\theta$  and  $k \in \mathbb{Z}$ ,  $(Pr \circ lm_\theta \circ p_{\theta,k})_* \beta_{\lambda,\mu}|_{N \times (T, \infty)}$  coincide with each other and in fact with  $\beta_{K,\lambda,\mu}$ . This completes the proof.

□3.2.

## 4 Natural Symplectic Structure on the Fermat Cubic Surface

### 4.1 Statements and notations

In this section, we prove the following fact on the symplectic structures of  $F_1$  induced from  $\mathbb{C}^3$ , which is the starting point of our main construction in the next section.

**Theorem 4.1** For a sufficiently large radius  $R_s \gg 1$  there exists a symplectic form  $\beta_1$  on the Fermat cubic surface  $F_1$  which satisfies the following properties.

- (1) On the end  $F_1 \cap \{\tau > T_s = 2 \log R_s\}$ , in the product coordinate  $N \times (T, \infty)$ ,  $\beta_1|_{N \times (T_s, \infty)} = d(e^\tau \zeta_N)$ .
- (2)  $\beta_1$  is invariant under the Hopf action  $H(t)$  for  $t \in 2\pi\mathbb{Z}/3$ .

**Corollary 4.2** For a sufficiently large  $T \gg 0$ , there exists a symplectic form  $\beta'$  on  $\mathbb{CP}^2 \setminus E_\omega$  whose restriction to the the product end  $U \setminus E_\omega$  satisfies  $\beta'|_{N' \times (T, \infty)} = d(e^\tau \zeta_{N'})$  with respect to the product coordinate  $N' \times (T, \infty)$ .

We explain some notations on  $\mathbb{CP}^2 \setminus E_\omega$ . First, for  $N'$ , see Remark 1.4. The natural contact 1-form  $\zeta_{N'}$  is obtained as  $\bar{\zeta}$  in Remark 1.3 and is also obtained as the quotient  $(N', \zeta_{N'}) = (N, \zeta_N)/\mathbb{Z}/3$ . As  $\mathbb{CP}^2 \setminus E_\omega$  is regarded as the quotient of  $F_1$  by the restricted Hopf action by  $\mathbb{Z}/3$ , on its end the product coordinate  $\cong N' \times (T, \infty)$  is also naturally induced from the product coordinate  $(T, \infty) \times N$  for the end of  $F_1$  by simply regarding  $N' = N/\mathbb{Z}/3$ .

We present two proofs of this theorem. The first one is simpler and is still sufficient for the purpose of this article, while the second one gives a slightly more accurate result which is a kind of stability.

### 4.2 Re-embedding of Fermat cubic surface

The first proof is to show the following proposition.

**Proposition 4.3** For a sufficiently large constant  $R_s \gg 1$  there exists a smooth embedding  $e_1$  of  $F_1$  into  $\mathbb{C}^3$  satisfying the following conditions.

- (1)  $e_1(F_1)$  is a symplectic submanifold of  $(\mathbb{C}^3, \beta^*)$ .
- (2)  $e_1|_{F_1 \cap \{\rho \geq R_s\}}$  coincides with the projection of the tubular neighbourhood of  $\check{F}_0 \cap \{\rho \geq R_s\}$ . In particular,  $e_1(F_1 \cap \{\rho \geq R_s\}) = F_0 \cap \{\rho \geq R_s\}$ .
- (3)  $e_1$  is equivariant under the Hopf action  $H(t)$  for  $t \in 2\pi\mathbb{Z}/3$ .

In fact, taking  $R_s$  larger, we can take  $e_1$  arbitrary  $C^1$ -close to the inclusion  $\iota_{F_1} : F_1 \hookrightarrow \mathbb{C}^3$  and they differ only on  $F_1 \cap \{e^{-1}R_s < \rho\}$ .

**Remark 4.4** In the above Proposition, the  $C^0$ - and the  $C^1$ -distances are measured based on the standard metric of  $\mathbb{C}^3$  on the target side and on the product coordinates on the source side. In what follows we prove it in a smaller  $C^1$ -distance, but it is not difficult to refine it to hold still in the standard metric.

*Proof of Proposition 4.3.* The ends of  $F_1$  and  $F_0$  are getting closer when  $\rho$  tends to  $\infty$ . Intuitively this is the reason for the proposition. Let us begin with formulating this fact as follows. It also serves the next subsection as a foundation.

In this article we fixed the Kodaira-Thurston nil-manifold as the quotient of  $\check{F}_0$  by homotheties of the scalar multiplication by  $e^{\pi\mathbb{Z}}$ . According to this, (even though we do not have to follow this at all,) we observe that the part  $F_1(R)$  of  $F_1$  converges to the part  $\check{F}_0(R)$  when  $R$  tends to  $\infty$  in the following sense. Here we introduced a new notation;  $F_w(R) = F_w \cap \{R \leq \rho \leq e^\pi R\}$ .

The homothety of  $\mathbb{C}^3$  by the scalar multiple by  $R$  pulls back both of the standard symplectic structure  $\beta^*$  and the standard Liouville form  $\lambda$  to their scalar multiple by  $R^2$ . Therefore it preserves especially the open subset  $\{4\text{-dimensional symplectic subspace}\}$  of the oriented Grassmannian  $\widetilde{Gr}(6,4)$  associated with each tangent space.

Respecting this property, we measure the distance between  $F_1(R)$  and  $\check{F}_0(R)$  after performing the scalar multiplication by  $R^{-1}$ , namely, by looking at  $R^{-1}F_1(R)$  and  $R^{-1}\check{F}_0(R) = \check{F}_0(1)$ . Also remark that  $R^{-1}F_1(R)$  exactly coincides with  $F_{R^{-3}}(1)$ .

The product structure  $\check{F}_0 \times D_\varepsilon^2$  of the tubular neighbourhood of  $\check{F}_0$  fixed in Section 3 enables us to regard  $F_w(1)$  as the graph of a smooth (analytic) function  $\hat{f}_w$  on  $\check{F}_0(1)$  if  $|w| \ll 1$ . The inverse function theorem implies the following, among which the  $C^1$ -convergence is sufficient for Proposition 4.3.

**Proposition 4.5** For any integer  $n \geq 0$ , the functions  $\hat{f}_w$  converge to 0 when  $|w|$  tends to 0 in any  $C^n$ -topology.

In other words, for any  $n$ , taking  $R$  larger,  $F_{R^{-3}}(1)$  is getting arbitrarily  $C^n$ -close to  $\check{F}_0(1)$ .

Now let us complete the proof of Proposition 4.3. For  $\check{F}_0$  here we adopt the polar coordinate  $(\rho, P) \in \mathbb{R}_+ \times N$ , so that  $\hat{f}_w$  is a function on  $\rho$  and  $P \in N$ . Choose and fix a smooth function  $\psi : [1, e^\pi] \rightarrow [0, 1]$  satisfying  $\psi(l) \equiv 1$  for  $l \leq 2$  and  $\psi(l) \equiv 0$  for  $l \geq 3$  and define  $f_w^+ = \psi(\rho)\hat{f}_w$ . Then it is easy to see that for  $|w| \ll 1$  the graph  $F_w^+(1)$  of  $f_w^+$  is  $C^1$ -close to  $\check{F}_0(1)$  and in fact becomes a symplectic submanifold of  $(\mathbb{C}^3, \beta^*)$ . Namely

for a sufficiently large  $R$ ,  $R(F_{R-3}^+(1))$  is also symplectic with respect to  $\beta^*$ . Taking  $R_s = e^\pi R$  we can define a new symplectic embedding  $e_1 : F_1 \hookrightarrow \mathbb{C}^3$  as

$$e_1|_{F_1 \cap \{\rho \leq eR\}} = \text{Id}_{F_1 \cap \{\rho \leq eR\}},$$

$$e_1(F_1 \cap \{R \leq \rho \leq R_s\}) = R(F_{R-3}^+(1)), \text{ and}$$

$$e_1|_{F_1 \cap \{e^2 R \leq \rho\}} = \text{the projection of the tubular neighbourhood of } \check{F}_0.$$

On  $F_1 \cap \{R \leq \rho \leq R_s\}$  as a map  $e_1$  is defined through the product structure of the tubular neighbourhood of  $\check{F}_0$ . The construction automatically fulfills the equivariance (3).  $\square$  4.3.

### 4.3 Stability of convex symplectic structure

As an alternative proof of Theorem 4.1, we show the following stability result, for which we need an aid of contact structure. This result must have already been intrinsically (or even extrinsically) used in many other articles and must be a kind of folklore.

**Theorem 4.6** For a sufficiently large constant  $T_s \gg 0$ , there exists an isotopy  $\Phi_s : F_1 \rightarrow F_1$  of  $F_1$  for  $s \in [0, 1]$  satisfying the following properties.

- (1)  $\Phi_0 = \text{Id}_{F_1}$ .
- (2)  $\Phi_1^*(\beta^*|_{F_1})|_{F_1 \cap \{\tau \geq T_s + 4\}} = d(e^\tau \zeta_N)$  in the product coordinate  $N \times (T_s, \infty)$  of the end of  $F_1$ .
- (3)  $\Phi_s$  is supported on  $\{\tau > T_s\}$  for any  $s \in [0, 1]$ .
- (4)  $\Phi_s$  is equivariant under the Hopf action  $H(t)$  for  $t \in 2\pi\mathbb{Z}/3$  for any  $s \in [0, 1]$ .

From the construction, it is naturally understood that taking  $T$  larger we can arrange the isotopy  $\Phi_s$  arbitrarily  $C^1$ -close to the trivial one. In what follows we focus our explanations rather on the fact that  $\beta^*|_{F_1}$  really coincides with  $\beta^*|_{\check{F}_0}$  on their ends through a suitable identification. The equivariance of the isotopy and its  $C^1$ -control are achieved by usual elementary argument so that they are not exactly discussed.

Here we use a convergence in the  $C^2$ -topology in Proposition 4.5 to obtain a  $C^1$ -convergence of contact structures. Let us begin with the review of the following well-known fact on the symplectization of contact structures.

**Proposition 4.7** Let  $W \subset \mathbb{C}^N$  be a holomorphically and properly embedded complex submanifold of  $\mathbb{C}^N \ni (Z_0, \dots, Z_N)$  for some  $N \in \mathbb{N}$ . Further we assume that the square  $\bar{\rho} = |Z_0|^2 + \dots + |Z_N|^2$  of the radius restricted to  $W$  has no critical points in  $\bar{\rho} \geq \bar{R}_*$ . Then  $(W^{\bar{R}^*} = W \cap \{\bar{\rho} \geq \bar{R}_*\}, \beta^*|_{W^{\bar{R}^*}})$  is symplectomorphic to the upper half of the standard symplectization of the contact manifold  $(M_{\bar{R}^*} = W \cap \{\bar{\rho} = R_*\}, \lambda^*|_{M_{\bar{R}^*}})$  where

$$\beta^* = 2 \sum_{j=0}^{N-1} dx_j \wedge dy_j, \quad \lambda^* = \sum_{j=0}^{N-1} (x_j dy_j - y_j dx_j), \quad Z_j = x_j + \sqrt{-1} y_j$$

are the standard ones. Described as in the end of Section 2, we also have  $\lambda^* = \bar{\rho} \zeta$  in the polar coordinate where  $\zeta = \lambda^*|_{S^{2N-1}}$ .

*Proof.* On  $W$  we have  $d(\lambda^*|_W) = \beta^*|_W$  from which we know that the vector field  $X$  defined as  $\iota_X \beta^*|_W = \lambda^*|_W$  satisfies  $\mathcal{L}_X \beta^*|_W = \beta^*|_W$ . Then it is easy to see that on  $\bar{\rho} \geq \bar{R}_*$   $X$  is non-singular,  $X \cdot \bar{\rho} > 0$ , of at most linear growth and therefore complete. Then the identification

$$\Psi : [1, \infty) \times M_{\bar{R}^*} \rightarrow W^{\bar{R}^*}, \quad \Psi(\varrho, P) = \exp(\log \varrho X)(P)$$

is a diffeomorphism and we have

$$\Psi^*(\lambda^*|_{W^{\bar{R}^*}}) = \varrho(\lambda^*|_{M_{\bar{R}^*}}), \quad \Psi^*(\beta^*|_{W^{\bar{R}^*}}) = d(\varrho(\lambda^*|_{M_{\bar{R}^*}})), \quad \text{and} \quad \Psi_* \varrho \frac{\partial}{\partial \varrho} = X.$$

□ 4.7.

In order to apply this proposition to  $F_1$  it is enough to take  $\bar{R}^* > 1$ . If we take  $\bar{R}^*$  larger, remark that  $\Psi$  extends even on the negative side of  $\log \varrho$ . To discuss in the product coordinate, let us change the coordinate  $\bar{\rho}$  into  $\tau = \log \bar{\rho} = 2 \log(\varrho)$ . Also we put  $T^* = \log \bar{R}^*$ . The following is a direct consequence of the above proposition.

**Corollary 4.8** For a constant  $\bar{R}^* > e$  there exists an isotopy  $\Phi[0]_s : F_1 \rightarrow F_1$  for  $s \in [0, 1]$  satisfying the following properties.

- (1)  $\Phi[0]_0 = \text{Id}_{F_1}$  and  $\Phi[0]_s$  is supported on  $\{\tau \geq T^* - 1\}$  for  $\forall s \in [0, 1]$ .
- (2)  $(\Phi[0]_1^*(\beta^*|_{F_1}))|_{\tau \geq T^*} = d(e^{\tau - T^*}(\lambda^*|_{F_1 \cap \{\tau = T^*\}}))$ , i.e., the symplectization of  $\lambda^*|_{F_1 \cap \{\tau = T^*\}}$ , where on the right hand side  $F_1 \cap \{\tau = T^*\}$  is identified with  $N = N \times \{T^*\}$  by the product coordinate.
- (3)  $\Phi[0]_s$  is equivariant under the Hopf action  $H(t)$  for  $t \in 2\pi\mathbb{Z}/3$  for any  $s \in [0, 1]$ .

*Proof of Theorem 4.6.* Based on these preparations, we have two steps to go. The first is an isotopy in the horizontal (i.e.,  $N$ ) direction and the second one is in  $\tau$  direction. Here we make every discussions on  $F_1 \cap \{\tau \geq T^* - 1\}$  in the product coordinate.

*Step I:* The convergence in  $C^2$ -topology in Proposition 4.5 implies that if we take  $\bar{R}^*$  large enough, the contact form  $\bar{R}^{*-2} \lambda^*|_{F_1 \cap \{\bar{\rho} = \bar{R}^*\}}$  on  $F_1 \cap \{\bar{\rho} = \bar{R}^*\} \cong N \times \{\tau = T^*\} \cong N$  becomes arbitrarily  $C^1$ -close to  $\zeta_N = \bar{R}^{*-2} \lambda^*|_{F_0 \cap \{\bar{\rho} = \bar{R}^*\}}$  on  $F_1 \cap \{\bar{\rho} = \bar{R}^*\} \cong N$ . Then we can connect the two contact forms by an segment in the space of contact 1-forms on  $N$  because in  $C^1$ -topology the space of contact forms on a closed manifold is an open subset of the space of all 1-forms. Especially the contact structure

$\ker [\bar{R}^{*-2} \lambda^*|_{F_1 \cap \{\bar{\rho}=\bar{R}^*\}}] = \ker [\lambda^*|_{F_1 \cap \{\bar{\rho}=\bar{R}^*\}}]$  on  $N$  is not only getting  $C^1$ -closer to but also as contact structures on  $N$  homotopic to  $\ker [\zeta_N]$  when  $\bar{R}^* \rightarrow \infty$ . Then Gray's stability theorem ([Gr]) implies that they are isotopic to each other. More precisely, there exists an isotopy  $\Phi[1; T^*]_s$  of  $N$  ( $s \in [0, 1]$ ) satisfying the followings.

- (1)  $\Phi[1; T^*]_0 = \text{Id}_N$ ,  $(\Phi[1; T^*]_1)^* \left( \ker [\lambda^*|_{F_1 \cap \{\bar{\rho}=\bar{R}^*\}}] \right) = \ker [\zeta_N]$ ,  
and  
 $(\Phi[1; T^*]_s)^* \left( \bar{R}^{*-2} \lambda^*|_{F_1 \cap \{\bar{\rho}=\bar{R}^*\}} \right) = (1-s) \bar{R}^{*-2} \lambda^*|_{F_1 \cap \{\bar{\rho}=\bar{R}^*\}} + s \zeta_N$ .
- (2)  $\Phi[1; T^*]_s$  is equivariant under the Hopf action  $H(t)$  for  $t \in 2\pi\mathbb{Z}/3$ .

Remark here that if the segment of 1-forms is short the isotopy is  $C^1$ -small. This isotopy is easily extended to an equivariant isotopy  $\Phi[1]_s$  of  $N \times [T^* - 1, \infty)$  which is supported on  $N \times [T^* - 1, \infty)$  and satisfies  $\Phi[1]_s(P, \tau) = (\Phi[1; T^*]_s(P), \tau)$  for  $\tau \geq T^*$ . Now put  $T_s = T^* - 1$ .

*Step II:* Combining previous two isotopies, we obtain

$$((\Phi[1]_1 \circ \Phi[0]_1)^*(\lambda^*|_{F_1}))|_{N \times [T^*, \infty)} = e^{\tau + \eta(P)} \zeta_N$$

for some smooth function  $\eta$  on  $N$ . Put  $\eta^* = \max\{|\eta(P)|; P \in N\}$ . Remark that  $\eta \rightarrow 0$  when  $T^* \rightarrow \infty$ . Then easily we can find an equivariant isotopy  $\Phi[2]_s$  of  $F_1$  which is supported on  $N \times [T^*, \infty)$  and satisfies

$$\Phi[2]_0 = \text{Id}_{F_1} \quad \text{and} \quad \Phi[2]_s(P, \tau) = (P, \tau - \eta(P)) \quad \text{for } \tau \geq T^* + \eta^* + 2.$$

Finally with all three isotopies, as a result we obtain the coincidence

$$((\Phi[2]_1 \circ \Phi[1]_1 \circ \Phi[0]_1)^*(\beta^*|_{F_1}))|_{N \times [T^*, \infty)} = d(e^\tau \zeta_N)$$

for  $\tau \geq T_s + 4 \geq T^* + 2\eta^* + 2$ .

The equivariance (3) is achieved by doing the above arguments on the quotient, because it is not only that the  $\mathbb{Z}/3$ -action preserves all the needed data but that it is a free action.  $\square$  4.6.

**Remark 4.9** Theorem 4.6 is generalized to many cases. Take a polynomial  $g(Z_0, \dots, Z_n)$  in  $(n+1)$ -variables and consider the hypersurface  $G_w = g^{-1}(w) \subset \mathbb{C}^{n+1}$  for  $w \in \mathbb{C}$ .  $G_w$  inherits a symplectic structures from  $\mathbb{C}^{n+1}$ . Then the structure on the end must be independent of  $w \in \mathbb{C}$  in the sense of Theorem 4.6. For (weighted) homogeneous polynomials, the result is generalized fairly directly. Further generalization to other polynomials or systems is to be pursued.

## 5 End-Periodic Symplectic Form on the Fermat Cubic Surface

Up to the previous section, we made detailed preliminaries but they are somehow trivial and simply supporting the discussions in this section. We prove the following theorem, again for which the arguments are surprisingly simple and easy. For simplicity, in this section, we use  $\tau$  for the first coordinate of the product end  $U \setminus E_\omega \cong N' \times (-\frac{2}{3} \log \varepsilon, \infty)$  of  $\mathbb{CP}^2 \setminus E_\omega$ .

**Theorem 5.1** For a sufficiently small constant  $0 < \mu \ll 1$  and sufficiently large constants  $T \gg 1$  and  $\lambda \gg 1$ , there exists a symplectic form  $\beta'_{\lambda,\mu}$  on  $\mathbb{CP}^2 \setminus E_\omega$  which restricts to  $\lambda d\tau \wedge dx + \mu dy \wedge \zeta_{N'}$  on its end  $N' \times (T, \infty)$ .

**Corollary 5.2** For the same constants as above, there exists a symplectic form  $\beta_{\lambda,\mu}$  on the Fermat cubic surface  $F_1$  which restricts to  $\lambda d\tau \wedge dx + \mu dy \wedge \zeta_{N'}$  on its end  $N \times (T, \infty)$ .  $\beta_{\lambda,\mu}$  is invariant under the Hopf action of  $H(t)$  for  $t \in 2\pi\mathbb{Z}/3$ . On the end  $N \times (T, \infty)$  naturally  $\beta_{\lambda,\mu}$  admits more symmetries, namely, it is invariant under the translations in  $\tau$ -direction and also under the Hopf action  $H(t)$  for any  $t \in \mathbb{R}$ .

Apart from the main result Theorem 3.1 of this article, this result might be of an independent interest. In the final section, we will make a brief discussion on the generalization of this result, namely, the (non-)existence of an end-periodic symplectic structure on Stein or globally convex symplectic manifolds. In the rest of this section, we prove the above theorem.

**Lemma 5.3** There exists a closed 2-form  $\kappa$  on  $\mathbb{CP}^2 \setminus E_\omega$  which restricts to  $dy \wedge \zeta_{N'}$  on the product end.

*Proof of Lemma 5.3.* As  $dy \wedge \zeta_{N'}$  is closed, it defines a de Rham cohomology class  $[dy \wedge \zeta_{N'}] \in H^2(N') \cong \mathbb{R}^2$ . Let us look at the Meyer-Vietoris exact sequence for the cohomologies of  $\mathbb{CP}^2 = \overline{U} \cup (\mathbb{CP}^2 \setminus U)$ . It is easy to see that the inclusion  $N' \hookrightarrow \overline{U}$  induces a trivial map  $0 : H^2(U) \rightarrow H^2(N')$ . Therefore the fact  $H^3(\mathbb{CP}^2) = 0$  and the long exact sequence tell us that the inclusion to the other side induces a surjective homomorphism  $H^2(\mathbb{CP}^2 \setminus U) \rightarrow H^2(N')$ . This also implies that the closed 2-form  $dy \wedge \zeta_{N'}$  on the product end extends to the whole  $\mathbb{CP}^2 \setminus E_\omega$  as a closed 2-form  $\kappa$ .  $\square_{5.3}$

**Remark 5.4** The origin of  $F_0$  is an isolated singularity of type *simple elliptic*. Up to 3-fold branched covering  $U$  is orientation-reversing diffeomorphic to the minimal blowing up resolution of  $F_0$ . On the other hand,  $\mathbb{CP}^2 \setminus E_\omega$  is up to 3-fold covering biholomorphic to the Milnor fibre. The above lemma reflects the fact that the resolution and the Milnor fibre are quite different to each other. Such a phenomenon does not happen for *simple* singularities. For singularity theory, see for example [D].

Let us proceed to construct an end-periodic symplectic form on  $\mathbb{CP}^2 \setminus E_\omega$ . First take a positive constant  $\mu$  small enough so that  $\beta' + \mu dy \wedge \zeta_{N'}$  is still a symplectic form. On the product end  $\{\tau \geq T_s\}$ , from Corollary 4.2 we know  $\beta' = d(e^\tau \zeta_{N'}) = e^\tau d\tau \wedge \zeta_{N'} + e^\tau \frac{3}{2\pi} dx \wedge dy$ . This implies  $\beta' \wedge dy \wedge \zeta_{N'} = 0$  and hence we have  $(\beta' + \mu dy \wedge \zeta_{N'})^2 = \beta'^2$  on the product end. Therefore if we choose  $\mu$  small enough, we can assure that even on the compact core  $\mathbb{CP}^2 \setminus U$ , the closed 2-form  $\beta' + \mu dy \wedge \zeta_{N'}$  is non-degenerate. We fix such  $\mu$ .

Next take constants  $T_s < T_0 < T_1 < T_2 < T_3 = T$  and non-negative smooth functions  $k(\tau)$  and  $l(\tau)$  of  $\tau$  on  $[T_0, \infty)$  satisfying the following conditions:

$$\begin{aligned} k(\tau) &= e^\tau, & l(\tau) &\equiv 0 : & T_0 \leq \tau \leq T_1, \\ k'(\tau) &> 0, & l(\tau) &> 0 : & T_1 \leq \tau < T_2, \\ k(\tau) &> 0, & l(\tau) &\equiv \lambda : & T_2 \leq \tau \leq T_3, \\ k(\tau) &\equiv 0, & l(\tau) &\equiv \lambda : & T_3 \leq \tau. \end{aligned}$$

This is done as follows. First choose such a smooth function  $k$ . Then take a constant  $\lambda > 0$  satisfying  $\max\{-\frac{3k'(\tau)k(\tau)}{4\mu\pi}; T_2 \leq \tau \leq T_3\} < \lambda$ . Then it is easy to find a smooth function  $l(\tau)$  which satisfies all of the above conditions.

Now we are ready to construct an end-periodic symplectic form. First modify  $\beta'$  on the product end. We can define  $\beta'_\sharp$  as

$$\beta'_\sharp = \begin{cases} \beta' & \text{on } \mathbb{CP}^2 \setminus U, \\ d(k(\tau)\zeta_{N'}) + l(\tau)d\tau \wedge dx & \text{on } N' \times [T_0, \infty) \end{cases}$$

because two expressions of  $\beta'_\sharp$  coincide with each other on  $N' \times [T_0, T_1]$ . Finally we put  $\beta'_{\lambda,\mu} = \beta'_\sharp + \mu\kappa$ . This is the desired symplectic form on  $\mathbb{CP}^2 \setminus E_\omega$  because of the following reasons. First of all,  $\beta'_{\lambda,\mu}$  is closed and coincides with  $\lambda d\tau \wedge dx + \mu dy \wedge \zeta_{N'}$  on  $N' \times [T_3, \infty)$  and with  $\beta' + \mu dy \wedge \zeta_{N'}$  on  $\mathbb{CP}^2 \setminus U$ . Therefore it is non-degenerate on  $\mathbb{CP}^2 \setminus U$  as already remarked above. On the product end, as  $d(k(\tau)\zeta_{N'})$  and  $l(\tau)d\tau \wedge dx + \mu dy \wedge \zeta_{N'}$  do not interact at all under the exterior product, we have

$$(\beta'_{\lambda,\mu})^2 = \left( \frac{3k'(\tau)k(\tau)}{2\pi} + 2l(\tau)\mu \right) d\tau \wedge dx \wedge dy \wedge \zeta_{N'}.$$

Therefore  $\beta'_{\lambda,\mu}$  is non-degenerate on the product end as well. □5.1.

## 6 $\tilde{E}_7$ and $\tilde{E}_8$

Among simple elliptic singularities, the following three deformation classes  $\tilde{E}_l$  ( $l = 6, 7, 8$ ) are known to be realized as hypersurface singularities and



their links are isomorphic to  $Nil^3(-3)$ ,  $Nil^3(-2)$ , and to  $Nil^3(-1)$  respectively. They are defined by the following polynomials.

$$\begin{aligned} f_{\tilde{E}_6} &= Z_0^3 + Z_1^3 + Z_2^3 \quad (+\lambda Z_0 Z_1 Z_2) \\ f_{\tilde{E}_7} &= Z_0^4 + Z_1^4 + Z_2^2 \quad (+\lambda Z_0 Z_1 Z_2) \\ f_{\tilde{E}_8} &= Z_0^6 + Z_1^3 + Z_2^2 \quad (+\lambda Z_0 Z_1 Z_2) \end{aligned}$$

As the smooth topology of these objects does not depend on the choice of the constant  $\lambda$  (except for finitely many exceptional values), in this paper we ignore it and take it to be 0. Each of our constructions in this paper for the Fermat cubic, *i.e.*, the  $\tilde{E}_6$  polynomial, also works in the other two cases. In this section we verify this fact, by briefly reviewing the topology of these singularities. For basic facts about hypersurface singularities, the readers may refer to Milnor's seminal text book [M]. Also Dimca's book [D] provides more detailed and modern basic informations.

The notations in §1 are used and interpreted in parallel or slightly modified meanings according to the context, unless otherwise specified. For  $f = f_{\tilde{E}_l}$  ( $l = 7, 8$ ) the origin is an isolated and in fact unique singularity of the hypersurfaces  $F_0$ . Instead of scalar multiplication, we define the weighted homogeneous action of  $\lambda \in \mathbb{C}^\times$  on  $\mathbb{C}^3$  by  $\lambda \cdot (Z_0, Z_1, Z_2) = (\lambda^{w_0} Z_0, \lambda^{w_1} Z_1, \lambda^{w_2} Z_2)$  where the weight vector  $\mathbf{w} = (w_0, w_1, w_2)$  takes value  $(2, 1, 1)$  [*resp.*  $(3, 2, 1)$ ] for  $\tilde{E}_l$  ( $l = 7, 8$ ). By this action we have  $\lambda \cdot F_w = F_{\lambda^4 w}$  [*resp.*  $\lambda \cdot F_w = F_{\lambda^6 w}$ ] for  $l = 7$  [*resp.*  $l = 8$ ]. The weighted homogeneous action by positive real numbers  $\lambda \in \mathbb{R}_+$  plays the role of the euclidean homotheties in the Fermat cubic case. The action by unit complex numbers  $\lambda = e^{it}$  ( $t \in \mathbb{R}$ ) restricts to the action on  $S^5$  and is denoted by  $h(t)$  and called the *weighted Hopf action*. The quotient space  $P_{\mathbf{w}}^2$  of this weighted Hopf action is called the *weighted projective space*, which is a complex analytic orbifold. The quotient map  $h : S^5 \rightarrow P_{\mathbf{w}}^2$  is called the *weighted Hopf fibration*, which is a Siefert fibration. We take  $\mathbb{CP}^2 = \{[X_0 : X_1 : X_2]\}$  as a quotient of  $P_{\mathbf{w}}^2$  as follows. Define a map  $\Phi : \check{\mathbb{C}}^3 \rightarrow \mathbb{CP}^2$  as  $\Phi : (Z_0, Z_1, Z_2) \mapsto [X_0 : X_1 : X_2] = [Z_0^{d_0} : Z_1^{d_1} : Z_2^{d_2}]$  where  $(d_0, d_1, d_2) = (1, 2, 2)$  [*resp.*  $(2, 3, 6)$ ] for  $\tilde{E}_7$  [*resp.*  $\tilde{E}_8$ ]. Then  $\Phi$  factors into  $\Phi|_{S^5} = \Psi \circ h$  for some  $\Psi : P_{\mathbf{w}}^2 \rightarrow \mathbb{CP}^2$ . The homogeneous equations

$$\begin{aligned} g_{\tilde{E}_7} &= X_0^2 + X_1^2 + X_2^2 = 0 \\ g_{\tilde{E}_8} &= X_0 + X_1 + X_2 = 0 \end{aligned}$$

on  $\mathbb{CP}^2$  rewrites  $f_{\tilde{E}_l} = 0$  as  $g_{\tilde{E}_l} \circ \Phi = 0$  ( $l = 7, 8$ ).

The first important fact to notice is that the open set of  $S^5$  consisting of all regular orbits of the weighted Hopf action  $h$  contains  $N = F_0 \cap S^5$ . Therefore the orbit space  $E_{(l)} = N / \sim_h$  is a non-singular holomorphic curve which sits in the regular part of  $P_{\mathbf{w}}^2$ .

' $g_{\tilde{E}_7} = 0$ ' defines a non-singular projective curve of degree 2 and ' $g_{\tilde{E}_8} = 0$ ' a projective line. Both of them are biholomorphic to  $\mathbb{CP}^1$ . Comparing  $\Phi$  and  $h|_{E_{(l)}}$ , we easily see that  $\Psi|_{E_{(7)}} : E_{(7)} \rightarrow \{X_0^2 + X_1^2 + X_2^2 = 0\}$  is a 2-fold branched covering over the rational curve with 4 branched points  $\{X_1 = 0 \text{ or } X_2 = 0\} \cap \{X_0^2 + X_1^2 + X_2^2 = 0\}$  like the Weierstrass  $\wp$  function and  $E_{(7)}$  is seen to be an elliptic curve. In the case of  $\tilde{E}_{(8)}$ ,  $\Psi|_{E_{(8)}} : E_{(8)} \rightarrow \{X_0 + X_1 + X_2 = 0\}$  is a 6-fold branched covering, branching over 3 points  $\{X_0 = 0\}$ ,  $\{X_1 = 0\}$ , and  $\{X_2 = 0\}$  with branch indices 2, 3, and 6 respectively. From this we also see that  $E_{(8)}$  is an elliptic curve.

Similarly it is easy to see that the self-intersection (the  $c_1$  of the normal bundle) of  $E_{(7)}$  [*resp.*  $E_{(8)}$ ] in  $P_{\mathbf{w}}^2$  is 8 [*resp.* 6] and that the  $c_1$  of the weighted Hopf fibrations over  $E_{(7)}$  [*resp.*  $E_{(8)}$ ] is  $-2$  [*resp.*  $-1$ ].

Like in the case of  $\tilde{E}_{(6)}$ , in both of the other two cases the weighted projection  $\check{C}^3 \rightarrow S^5 = \check{C}^3/\mathbb{R}_+$  by positive real numbers restricts to a diffeomorphism from  $F_{re^{i\theta}}$  to the Milnor fibre  $L_\theta$ .  $h|_{L_\theta} : L_\theta \rightarrow P_{\mathbf{w}}^2 \setminus \tilde{E}_{(l)}$  is a branched covering, but the number of branched points is finite and around the ends it is a 4-fold [*resp.* 6-fold] regular covering for  $l = 7$  [*resp.*  $l = 8$ ].

We also remark here that the link  $N$  has a product type tubular neighbourhood  $W$  in  $P_{\mathbf{w}}^2$  because  $f$  gives the trivialization. The boundary  $\partial W$  is a Kodaira-Thurston nil-manifold and  $\tilde{F}_0$  can be considered as its cyclic covering.

Now let us verify that our constructions are transplanted to the cases of  $E_{(7)}$  and  $E_{(8)}$ . From the descriptions of the link  $N$ , the Milnor fibres  $L_\theta$ , and of  $F_1$ , the contents in Section 1 and 2 are recovered. Looking at the weighted homogeneous action of  $\mathbb{R}_+$ , we see  $F_1$  is approaching to  $\tilde{F}_0$  on the end and that the results in Section 4 hold in a parallel way.

As to the results in Section 5, once a parallel result to Lemma 5.3 is verified, then the manipulations of differential forms on the product end holds without major modifications. Together with the commutative diagram below, the fact that the rational (or real) cohomology of  $P_{\mathbf{w}}^2$  is isomorphic to that of  $\mathbb{CP}^2$  (see *e.g.*, [D]) tells that a parallel to Lemma 5.3 holds.

$$\begin{array}{ccccc} N \times (T, \infty) & \cong & \text{end of } \tilde{F}_0 \cong \text{end of } F_1 & \hookrightarrow & F_1 \cong L_0 \\ \downarrow & & \downarrow & & \downarrow h|_{L_0} \\ \partial U \times (T, \infty) & \cong & \text{end of } P_{\mathbf{w}}^2 \setminus E_{(l)} & \hookrightarrow & P_{\mathbf{w}}^2 \setminus E_{(l)} \end{array}$$

The left and the middle vertical arrows are regular coverings and the right one is a branched covering.

## 7 Concluding Remarks

To close the present article, we make some comments and raise some questions related to our construction.

## 7.1 End-periodic symplectic structures on Stein or globally convex symplectic manifolds

The construction of leafwise symplectic structure in this paper seems to stand on an extremely rare intersection of fortunes.

Besides the foliations of codimension one, as is mentioned in the previous section, the existence of end-periodic symplectic structures on Stein or globally convex symplectic manifolds might be of an independent interest. However, the possibility of such cases seems to be still limited. In the final section we discuss on such problems.

**Example 7.1** The Stein manifold  $\mathbb{C}$  (or the upper half plane  $\mathbb{H}$ ) carries an end-periodic symplectic form.

This example is in many senses trivial, because, first of all the fact itself is trivial. Especially we do not have to change the symplectic form. Also, as this Stein manifold is not really convex, we should say this is a meaningless example. The convexity of symplectic structures must be discussed on manifolds of dimension  $\geq 4$  (see [EG]). However, this example still exhibits a clear contrast to the following example.

**Example 7.2** The Stein manifold  $\mathbb{C}^n$  ( $n \geq 2$ ) does not admit an end-periodic symplectic structure, basically because  $S^1 \times S^{2n-1}$  does not admit such structures.

As the non-existence this example is generalized to many cases.

For the case of symplectic dimension 4, the recent result by Friedl and Vidussi, which has been known as Taubes' conjecture, provides a strong constraint.

**Theorem 7.3** (Taubes' conjecture, Friedl-Vidussi [FV]) For a closed 3-manifold  $M$ , the 4-manifold  $W^4 = S^1 \times M^3$  admits a symplectic structure if and only if  $M$  fibers over the circle.

Now in order to make the implication of this theorem clearer, let us take the following definition.

**Definition 7.4** Assume that an open  $2n$ -manifold  $W$  has an end which is diffeomorphic to  $\mathbb{R}_+ \times M^{2n-1}$  for some closed oriented manifold  $M$ . An *end-periodic symplectic structure* on  $W$  is a symplectic structure on  $W$  whose restriction to the end is invariant under the action of non-negative integers  $\mathbb{N}_0$  where  $n \in \mathbb{N}_0$  acts on  $\mathbb{R}_+ \times M$  as  $(t, m) \mapsto (t + n, \varphi^n(m))$  for some fixed monodromy diffeomorphism  $\varphi : M \rightarrow M$ .

It follows directly from the definition that the mapping cylinder  $M_\varphi$  admits a symplectic structure. If the monodromy belongs to a mapping class of finite order,  $S^1 \times M$  also admits a symplectic structure.

Above theorem due to Friedl and Vidussi tells that in the trivial or finite monodromy case, the  $M^3$  must fibres over the circle. Essentially the same construction in the case of the Kodaira-Thurston nil-manifold gives symplectic structures on such closed 4-manifolds. The virtue of their theorem is of course in the converse implication.

**Example 7.5** (Higher degree surfaces) Instead of taking the Fermat cubic surface as in the present article, if we take, *e.g.*, a Fermat type quartic surface, then the end is diffeomorphic to  $\mathbb{R}_+ \times M^3$ , where  $M^3$  is an  $S^1$ -bundle over the closed oriented surface  $\Sigma_3$  of genus 3 with euler class 4. As this 3-manifold apparently does not fiber over the circle, the Fermat quartic surface does not admit an end-periodic symplectic structure with at most finite monodromy. The same applies also to weighted homogeneous hypersurfaces  $\{Z_0^p + Z_1^q + Z_2^r = 1\}$  with  $1/p + 1/q + 1/r \neq 0$ .

If we extend our scope from Stein manifolds to globally convex symplectic manifolds (see [EG] for details of this notion) we find one more example for the existence of end-periodic symplectic structure, which is only slightly less trivial than Example 7.1.

**Example 7.6** (Solvable manifold) Let  $M^3 = \text{Sol}$  be a 3-dimensional compact solv-manifold, namely, the mapping cylinder of a hyperbolic automorphism of  $T^2$ .

Then it carries an algebraic Anosov flow, whose strong (un)stable direction corresponds to an eigenvalue. Then from [Mi] we know that  $\mathbb{R} \times M$  admits a globally convex symplectic structure. As it has a disconnected end, it is not Stein. On the other hand, apparently  $\mathbb{R} \times M$  admits an end-periodic symplectic structure because  $S^1 \times M$  is symplectic.

Some of higher genus surface bundles over the circle with pseudo-Anosov monodromy admit Anosov flows. See [Go] for Anosov flow on hyperbolic 3-manifolds. Of course in this formulation, Thurston's conjecture is involved, which asserts that any closed hyperbolic 3-manifold admits a finite covering which fibers over the circle. The above example is extended to those.

These examples are again disappointing because the compact core part has no more topology than the end.

Apart from trivial constructions like taking products of the Fermat cubic surfaces, Examples 7.1 and 7.6 are the only known other such manifolds that is convex and at the same time admits an end-periodic symplectic structure. Also remark that the construction of differential forms in Section 5 might have some similarities only on manifolds of dimension  $4k$  ( $k \in \mathbb{N}$ ).

## 7.2 Foliations on spheres

Meersseman and Verjovsky proved in [MV2] that Lawson's foliation does not admit a leafwise complex structure. The principal obstruction occurs in the tube component. It is interesting that comparing the tube component and the Fermat component, the difficulties in introducing leafwise complex structures are the other way round than in constructing leafwise symplectic structures. At present, the existence problem of a foliation of codimension-one on  $S^5$  with smooth leafwise complex structure is totally open.

Also the existence of codimension-one foliation with leafwise symplectic structures on higher odd dimensional spheres seems to be widely open. These problems might be more interesting if they are considered in relations with contact structures.

A recent result by G. Meigniez ([Me]) claims that in dimension  $\geq 4$ , once a smooth foliation of codimension one exists on a closed manifold, it can be modified into a minimal one, namely, one with every leaf dense. Especially it follows that in higher dimensions there is no more direct analogue of Novikov's theorem, that is, for any foliation of codimension 1 on  $S^3$  there exists a compact leaf. It is also mentioned by Meigniez that under the presence of some geometric structures, like leafwise symplectic structures, it might be of some interest to ask whether some similar statement to Novikov's theorem holds or not.

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